SUBGROUPS OF MOD(S) GENERATED BY $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$ AND $Y \in \{T_a, T_b\}$

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September 26, 2011

Abstract

Suppose a and b are distinct isotopy classes of essential simple closed curves in an orientable surface S. Let T_a and T_b represent the respective Dehn twists along a and b. In this paper, we study the subgroups of Mod(S) generated by X and Y, where $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$, $k \in \mathbb{Z}$, and $Y \in \{T_a, T_b\}$. For a large class of examples, we show that the subgroups $\langle X, Y\rangle$ and $\langle T_a, T_b\rangle$ are isomorphic. Moreover, we prove that $\langle X, Y\rangle = \langle T_a, T_b\rangle$ whenever $i(a, b) = 1$ and k is not a multiple of three or $i(a, b) \geq 2$ and $k = \pm 1$. Further, we compute the index $[\langle T_a, T_b\rangle : \langle X, Y\rangle]$ when $\langle X, Y\rangle$ is a proper subgroup of $\langle T_a, T_b\rangle$.

1 Introduction and Motivation

Let $S = S_{g,b}$ be a surface of genus g and b boundary components. Throughout this paper, we assume that S is connected, orientable, compact, and of finite type. Denote by Mod(S) the mapping class group of S, which is the group of isotopy classes of orientation preserving homeomorphisms of S which fix the boundary ∂S pointwise.

Let a and b represent distinct isotopy classes of essential simple closed curves in S, and let T_a and T_b be the respective Dehn twists along a and b. In this paper, we investigate the subgroups $\langle X, Y\rangle$ of Mod(S), where $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$, $k \in \mathbb{Z}$, and $Y \in \{T_a, T_b\}$. We compute $\langle X, Y\rangle$ based on k and the geometric intersection numbers $i(a, b)$. In particular, we show that $\langle X, Y\rangle$ is isomorphic to one of the following groups: $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathcal{B}_3$, $SL_2(\mathbb{Z})$, or $\mathbb{F}_2$. It turns out that $\langle X, Y\rangle \cong \langle T_a, T_b\rangle$ whenever $i(a, b) \neq 1$ and $k \neq 0$. Moreover, the two groups coincide whenever $i(a, b) = 1$ and k is not a multiple of three or $i(a, b) \geq 2$ and $k = \pm 1$. In the first case, the group generated by T_a and T_b is isomorphic to $SL_2(\mathbb{Z})$ if S is the torus $S_{1,0}$ and is isomorphic to the braid group $\mathcal{B}_3$ on three strands when $S \neq S_{1,0}$. In the second case, the group generated by T_a and T_b is isomorphic to the free group on two generators.

Consider two distinct isotopy classes a and b of essential simple closed curves in S. Denote by T_a and T_b the respective (left) Dehn twists along a and b. Let $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$, $k \in \mathbb{Z}$, and $Y \in \{T_a, T_b\}$. Denote by G the subgroup of Mod(S) generated by X and Y. The structure of G is independent of the isotopy classes a and b. Rather, G depends only on k and the geometric intersection $i(a, b)$. Since $i(a, b)$ is symmetric in a and b, it follows that the subgroups $\langle (T_aT_b)^k, T_a\rangle$ and $\langle (T_bT_a)^k, T_b\rangle$ are isomorphic. Similarly, $\langle (T_aT_b)^k, T_b\rangle$ and $\langle (T_bT_a)^k, T_a\rangle$ are isomorphic. Thus, the structure of G is symmetric with respect to T_a and T_b, and it is enough to study G modulo this symmetry. In other words, it suffices to consider $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$ and fix $Y = T_a$ in order to investigate the group G. We prove the following theorem:

Theorem 1.1 (Main Theorem)**.** *Suppose that a and b are distinct isotopy classes of essential simple closed curves in S. Let T_a and T_b denote the (left) Dehn twists along a and b respectively. Let $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$, $k \in \mathbb{Z}$, and $Y = T_a$. Denote by G the subgroups of Mod(S) generated by X and Y.*

- *If $k = 0$, then $G = \langle T_a\rangle \cong \mathbb{Z}$*
- *If $k \neq 0$ and $i(a, b) = 0$, then $G = \langle T_a, T_b^k\rangle \cong \mathbb{Z}^2$. Moreover, G has index k in $\langle T_a, T_b\rangle$.*
- *If $k \neq 0$ and $i(a, b) \geq 2$, then $G \cong \mathbb{F}_2$. Moreover, $G = \langle T_a, T_b\rangle$ when $k = \pm 1$ and G is a subgroup of infinite index in $\langle T_a, T_b\rangle$ otherwise.*
- *If $k \neq 0$ and $i(a, b) = 1$, then*

  When $S = S_{1,0}$,

$$G = \begin{cases} \langle T_a, T_b\rangle \cong SL_2(\mathbb{Z}) & \text{if } k \not\equiv 0 \; mod(3) \\ \mathbb{Z}_2 \times \mathbb{Z} & \text{if } k \equiv 3 \; mod(6) \\ \langle T_a\rangle \cong \mathbb{Z} & \text{if } k \equiv 0 \; mod(6) \end{cases}$$

  In the last two cases, G has infinite index in $\langle T_a, T_b\rangle$.

  When $S \neq S_{1,0}$,

$$G = \begin{cases} \langle T_a, T_b\rangle \cong \mathcal{B}_3 & \text{if } k \not\equiv 0 \; mod(3) \\ \mathbb{Z}^2 & \text{if } k \equiv 0 \; mod(3) \end{cases}$$

  In the second case, G has infinite index in $\langle T_a, T_b\rangle$.

Acknowledgements. I am very grateful to Sergio Fenley for carefully reading this paper and suggesting improvements.

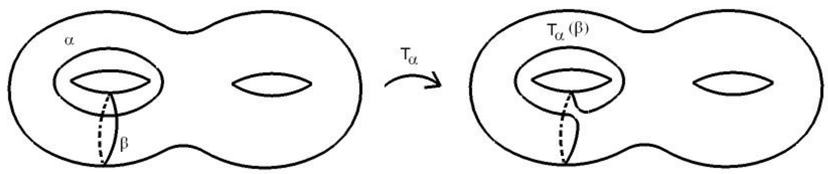

Figure 1: The effect of the Dehn twist T_α on the simple closed curve β.

2 Background on Dehn Twists

This section provides a basic background about Dehn twists and some of their relevant properties. For more information, the reader is referred to [4], [6], and [1].

Let α be a simple closed curve in S and let $N = N_\epsilon(\alpha)$ denote a regular neighborhood of α. A left Dehn twist (with respect to the orientation of S) along α is a homeomorphism $T_\alpha : S \to S$ which is supported on N and is the identity on the complement of N. If β is an arc transverse to α, then T_α affects β by causing it to turn left near the intersection point, go once around α, then proceed along β as before. See Figure 1 for an illustration.

Let a represent the isotopy class of α. The isotopy class (or mapping class) of the homeomorphism T_α is an element of Mod(S). We shall denote this element by T_a and refer to it as the (left) Dehn twist along a. In what follows, a Dehn twist will always mean an element of Mod(S).

If a and b are isotopy classes of simple closed curves in S, then the geometric intersection number $i(a, b)$ is the minimal number of intersection points between the representatives of a and b. That is,

$$i(a, b) = min_{\substack{\alpha \in a \\ \beta \in b}} |\alpha \cap \beta|$$

Let a and b represent isotopy classes of simple closed curves in S, and denote by T_a and T_b their respective (left) Dehn twists in Mod(S).

Fact 2.1. *$T_a = T_b \Leftrightarrow a = b$.*

Fact 2.2. *T_a has infinite order.*

Fact 2.3. *If $f \in Mod(S)$, then $fT_a f^{-1} = T_{f(a)}$.*

The following fact follows easily from Facts 2.1 and 2.3.

Fact 2.4. *Let $f \in Mod(S)$. Then $fT_a = T_a f \Leftrightarrow f(a) = a$.*

Fact 2.5. *If n is an integer, then $i(T_a^n(b), b) = |n| i(a, b)^2$.*

Fact 2.6. *$T_a T_b = T_b T_a \Leftrightarrow i(a, b) = 0$. The left hand side of the equivalence is called the commutativity of disjointness relation.*

Fact 2.7. *If a and b are distinct, then $T_a T_b T_a = T_b T_a T_b \Leftrightarrow i(a, b) = 1$. The left hand side of the equivalence is known as the braid relation.*

The following fact is an easy consequence of Facts 2.1, 2.3, and 2.7.

Fact 2.8. *If $i(a, b) = 1$, then $T_a T_b(a) = b$ and $T_b T_a(b) = a$.*

Fact 2.9. *If $a \neq b$ and $T_a^p = T_b^q$ for some $p, q \in \mathbb{Z}$, then $p = q = 0$.*

Theorem 2.10 (2-Chain Relation)**.** *If $i(a, b) = 1$, then $(T_a T_b)^6 = T_c$ in Mod(S), where c is the boundary of a closed regular neighborhood of $a \cup b$. In particular, T_c is trivial when $S = S_{1,0}$ and so $(T_a T_b)^6 = 1$ in this case.*

Theorem 2.11. *Denote by Γ the subgroup of Mod(S) generated by T_a and T_b. Then*

- $\Gamma \cong \mathbb{Z}^2$ *if $i(a, b) = 0$*
- $\Gamma \cong SL_2(\mathbb{Z})$ *if $i(a, b) = 1$ and $S = S_{1,0}$*
- $\Gamma \cong \mathcal{B}_3$ *if $i(a, b) = 1$ and $S \neq S_{1,0}$*
- $\Gamma \cong \mathbb{F}_2$ *if $i(a, b) \geq 2$ (Ishida [5])*

3 Group Theory

Given a free group on finitely many generators and a finite index subgroup H, the following theorem (Theorem 2.10 in [3]), implies that H is a free group and determines the number of its generators.

Theorem 3.1. *Consider the free group $\mathbb{F}_p$ on p generators and let H be an index q subgroup in $\mathbb{F}_p$. If p and q are finite, then H is a free group on $q(p-1)+1$ generators.*

Theorem 3.2. *Suppose that G is a virtually abelian group and let H be a subgroup of G. Then H is virtually abelian*

Proof. G has a finite index subgroup K which is abelian. Since $[H : H \cap K] \leq [G : K]$, it follows that $H \cap K$ has finite index in H. Moreover, $H \cap K$ is abelian as it is a subgroup of K. Therefore, H is virtually abelian. $\square$

Corollary 3.3. *The braid group $\mathcal{B}_3$ is not virtually abelian.*

Proof. Choose a and b in $S \neq S_{1,0}$ so that $i(a,b) = 1$. By Theorem 2.11, the group generated by T_a and T_b is isomorphic to $\mathcal{B}_3$. By Fact 2.5, $i(T_a^2(b), b) = 2$, and it follows from Theorem 2.11 that $T_{T_a^2(b)}$ and T_b generate the free group $\mathbb{F}_2$ of order two. Since $\mathbb{F}_2$ is not virtually abelian, $\langle T_a, T_b \rangle$, and consequently $\mathcal{B}_3$, is not virtually abelian. $\square$

Corollary 3.4. *The modular group $SL_2(Z)$ is not virtually abelian.*

Proof. Choose a and b in $S = S_{1,0}$ so that $i(a,b) = 1$. T_a and T_b generate Mod(S), which is known isomorphic to $SL_2(Z)$ (see [4]). As in the proof of Corollary 3.3, $T_{T_a^2(b)}$ and T_b generate $\mathbb{F}_2$, which not virtually abelian. Therefore, $SL_2(Z)$ is not virtually abelian. $\square$

4 Proof of the Main Theorem

Recall that $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$, $k \in \mathbb{Z}$, and $Y = T_a$.

If $k = 0$, then X is trivial and $G = \langle X, Y \rangle = \langle Y \rangle$. By Fact 2.2, Y has infinite order, and so $G = \langle T_a \rangle \cong \mathbb{Z}$.

Assume $k \neq 0$.

If $i(a,b) = 0$, then $\langle T_a, T_b \rangle \cong \mathbb{Z}^2$ by Theorem 2.11. In particular, $X = T_a^k T_b^k$ and hence, $G = \langle T_a, T_b^k \rangle$. Since T_a and T_b commute, every element in G can be expressed in the form $T_a^\alpha T_b^\beta$ for some $\alpha, \beta \in \mathbb{Z}$. If $T_a^\alpha T_b^\beta = 1$, it follows from Fact 2.9 that $\alpha = \beta = 0$. As such, G is torsion free. This fact combined with $[X, Y] = 1$ imply that $G \cong \mathbb{Z}^2$. Moreover, $\langle T_a, T_b \rangle / \langle T_a, T_b^k \rangle \cong \mathbb{Z}_k$ and so $G = \langle T_a, T_b^k \rangle$ has index k in $\langle T_a, T_b \rangle$.

Now suppose that $i(a,b) \geq 2$. By Theorem 2.11, $\langle T_a, T_b \rangle \cong \mathbb{F}_2$. Since G is a two generator subgroup of the free group $\mathbb{F}_2$, G is either infinite cyclic or free on two generators. As the generators X and Y of G have no common powers, G is isomorphic to $\mathbb{F}_2$. If $k = \pm 1$, then $G = \langle T_a, (T_aT_b)^{\pm 1} \rangle = \langle T_a, T_b \rangle$. Now assume $k \neq \pm 1$. If G has finite index q in $\langle T_a, T_b \rangle$, then $q = 1$ by Theorem 3.1. But $\langle T_a, T_b \rangle = \langle T_a, T_aT_b \rangle$ and $\langle T_a, T_aT_b \rangle$ modulo the normal closure of $\langle T_a, (T_aT_b)^k \rangle$ is isomorphic to the cyclic group $\mathbb{Z}_k$ of order k. As such, G is a proper subgroup in $\langle T_a, T_b \rangle$ and so $q \neq 1$; a contradiction. Therefore, G has infinite index in $\langle T_a, T_b \rangle$. The case when $X = (T_bT_a)^k$ and $Y = T_a$ follows similarly.

As shown above, note that G and $\langle T_a, T_b\rangle$ are isomorphic groups when $i(a,b) \neq 1$ and $k \neq 0$. More precisely, G and $\langle T_a, T_b\rangle$ are both isomorphic to $\mathbb{Z}^2$ when $i(a,b) = 0$ and $k \neq 0$, and both groups are isomorphic to $\mathbb{F}_2$ when $i(a,b) \geq 2$ and $k \neq 0$.

The proof of the Main Theorem when $i(a,b) = 1$ is done through Lemma 4.1, Proposition 4.2, and Lemma 4.3. Lemma 4.1 shows that conjugating Y with X depends primarily on k and the conjugation can be easily determined once we specify the residue of k modulo three. Proposition 4.2 shows that G is equal to $\langle T_a, T_b\rangle$ whenever $i(a,b) = 1$ and k is not a multiple of three. Finally, Lemma 4.3 investigates the structure of G when k is a multiple of three.

Lemma 4.1. *Let k be a positive integer and suppose that a and b are isotopy classes such that $i(a,b) = 1$. Then*

	$k \equiv 0\ mod(3)$	$k \equiv 1\ mod(3)$	$k \equiv 2\ mod(3)$
$(T_aT_b)^kT_a(T_aT_b)^{-k}$	T_a	T_b	$T_aT_bT_a^{-1}$
$(T_aT_b)^{-k}T_a(T_aT_b)^{k}$	T_a	$T_aT_bT_a^{-1}$	T_b
$(T_bT_a)^kT_a(T_bT_a)^{-k}$	T_a	$T_bT_aT_b^{-1}$	T_b
$(T_bT_a)^{-k}T_a(T_bT_a)^{k}$	T_a	T_b	$T_a^{-1}T_bT_a$

Proof. We only prove the first row of Table 4.1. The remaining rows can be shown similarly. We proceed by induction on k.

$$\begin{aligned} k = 1 : (T_aT_b)T_a(T_aT_b)^{-1} &= T_aT_bT_aT_b^{-1}T_a^{-1} \\ &= T_bT_aT_bT_b^{-1}T_a^{-1} \\ &= T_b \end{aligned}$$

$$\begin{aligned} k = 2 : (T_aT_b)^2T_a(T_aT_b)^{-2} &= (T_aT_b)T_b(T_aT_b)^{-1} \\ &= T_aT_bT_a^{-1} \end{aligned}$$

$$\begin{aligned} k = 3 : (T_aT_b)^3T_a(T_aT_b)^{-3} &= (T_aT_b)(T_aT_bT_a^{-1})(T_aT_b)^{-1} \\ &= T_aT_bT_b^{-1}T_aT_bT_b^{-1}T_a^{-1} \\ &= T_a \end{aligned}$$

where the third equality is a consequence of the braid relation.

This takes care of the base case. Assume that the first row holds for some $k \geq 4$. Then

$$\begin{aligned} (T_aT_b)^{k+1}T_a(T_aT_b)^{-(k+1)} &= (T_aT_b)(T_aT_b)^kT_a(T_aT_b)^{-k}(T_aT_b)^{-1} \\ &= \begin{cases} T_b & \text{if } k \equiv 0\ mod(3) \\ T_aT_bT_a^{-1} & \text{if } k \equiv 1\ mod(3) \\ T_a & \text{if } k \equiv 2\ mod(3) \end{cases} \end{aligned}$$

□

Proposition 4.2. *Suppose that a and b are isotopy classes of simple closed curves in S such that $i(a,b)=1$. If $k \not\equiv 0\ mod(3)$, then $G = \langle T_a, T_b \rangle$. By Theorem 2.11, this implies that $G \cong SL_2(\mathbb{Z})$ when $S = S_{1,0}$ and $G \cong \mathcal{B}_3$ when $S \neq S_{1,0}$.*

Proof. Recall that $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$, $k \in \mathbb{Z}$, and $Y = T_a$. Assume that $i(a,b) = 1$ and $k \not\equiv 0\ mod(3)$. The following table shows how to generate all of $\langle T_a, T_b \rangle$ from X and Y. More precisely, the table indicates how to obtain T_b from X and $Y = T_a$. This implies that $G = \langle X, Y \rangle = \langle T_a, T_b \rangle$. For example, if $X = (T_aT_b)^k$, $Y = T_a$, $k > 0$, and $k \equiv 1\ mod(3)$, then $XYX^{-1} = T_b$ according to the table below. That $XYX^{-1} = T_b$ follows from Lemma 4.1. If, on the other hand, $X = (T_aT_b)^k$, $Y = T_a$, $k > 0$, and $k \equiv 2\ mod(3)$, then $Y^{-1}XYX^{-1}Y = T_b$ according to the table. To see why $Y^{-1}XYX^{-1}Y = T_b$, note that $XYX^{-1} = T_aT_bT_a^{-1}$ by Lemma 4.1 and recall that $Y = T_a$. The remaining entries in the table below can be checked in a similar fashion.

X	Y	$k \equiv 1\ mod(3)$	$k \equiv 2\ mod(3)$
$(T_aT_b)^k$	T_a	XYX^{-1}	$Y^{-1}XYX^{-1}Y$
$(T_aT_b)^{-k}$	T_a	$Y^{-1}XYX^{-1}Y$	XYX^{-1}
$(T_bT_a)^k$	T_a	$YXYX^{-1}Y^{-1}$	XYX^{-1}
$(T_bT_a)^{-k}$	T_a	XYX^{-1}	$YXYX^{-1}Y^{-1}$

□

Lemma 4.3. *Consider a and b such that $i(a,b) = 1$.*

- *If $S \neq S_{1,0}$ and $k \equiv 0\ mod(3)$, then $G \cong \mathbb{Z}^2$.*
- *If $S = S_{1,0}$ and $k \equiv 0\ mod(6)$, then $G = \langle T_a \rangle \cong \mathbb{Z}$.*
- *If $S = S_{1,0}$ and $k \equiv 3\ mod(6)$, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}$.*

Proof. First assume that $i(a,b) = 1$ in $S \neq S_{1,0}$. By Theorem 2.11, $\langle T_a, T_b \rangle \cong \mathcal{B}_3$. It is well known [7] that the center of $\langle T_a, T_b \rangle$ is infinite cyclic, generated by $(T_aT_b)^3$. Moreover, it is an immediate consequence of the braid relation that

$$(T_aT_b)^3 = (T_bT_a)^3 \qquad (*)$$

So $(T_bT_a)^3$ generates the center of $\langle T_a, T_b \rangle$ as well. As such, $[X,Y] = 1$ for all $X \in \{(T_aT_b)^k, (T_bT_a)^k\}$, $k = 3n$ with $n \neq 0$, and $Y = T_a$. So every element in G can be expressed in the form $X^\alpha Y^\beta$ for some $\alpha, \beta \in \mathbb{Z}$. For all nonzero β, note that $X^\alpha Y^\beta(b) \neq b$. To see this, observe that X^α is central in $\langle T_a, T_b \rangle$ and thus commutes with both Y^β and T_b. As such, $X^\alpha Y^\beta(b) = Y^\beta X^\alpha(b) = Y^\beta(b)$ where the last equality is due to Fact 2.4. Moreover, since $i(Y^\beta(b), b) = |\beta| i(a,b)^2 = |\beta| > 0$,

$Y^(\beta) \neq b$. $X^\alpha Y^\beta(b) \neq b$ combined with the fact that $\langle X \rangle$ is infinite cyclic imply that G is torsion free. Consequently, $G \cong \mathbb{Z}^2$. Further, since $\langle T_a, T_b \rangle \cong \mathcal{B}_3$ is not virtually abelian (by Corollary 3.3), the abelian subgroup G must have infinite index.

Now assume that $i(a, b) = 1$ in the torus $S = S_{1,0}$. By Theorem 2.11, $\langle T_a, T_b \rangle \cong SL_2(\mathbb{Z})$. It still follows from the braid relation that $(*)$ holds. It is easy to check that $(T_a T_b)^3$ is a nontrivial mapping class. Moreover, it follows from Theorem 2.10 that $(T_a T_b)^6 = 1$. As such, $(T_a T_b)^3$ has order two. Thus, X equals the identity when $k \equiv 0 \, mod(6)$ and is an involution when $k \equiv 3 \, mod(6)$. In the first case, it is immediate that $G = \langle Y \rangle \cong \mathbb{Z}$. In the second case, X is an order two element which commutes with the infinite order element Y. Therefore, $G \cong \mathbb{Z}_2 \times \mathbb{Z}$. Finally, Corollary 3.4 implies that $\langle T_a, T_b \rangle \cong SL_2(\mathbb{Z})$ is not virtually abelian. As such, the abelian subgroups G that are isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2 \times \mathbb{Z}$ cannot have finite index. $\square$

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